Modern Physics:Lecture 08: Harmonic Oscillator

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Why Study the Quantum Harmonic Oscillator?

- **Ubiquitous model**: Appears in molecular vibrations, quantum optics, solid-state physics
- **Exactly solvable**: Demonstrates quantization naturally
- **Foundational**: Basis for quantum field theory and many-body systems
- **Experimental relevance**: Laser cooling, trapped ions, nanomechanical oscillators

Classical Harmonic Oscillator

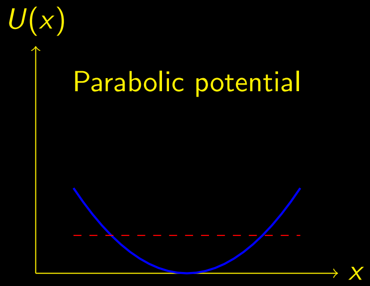
$$F = -kx$$

$$m\ddot{x} = -kx$$

$$\omega = \sqrt{\frac{k}{m}}$$

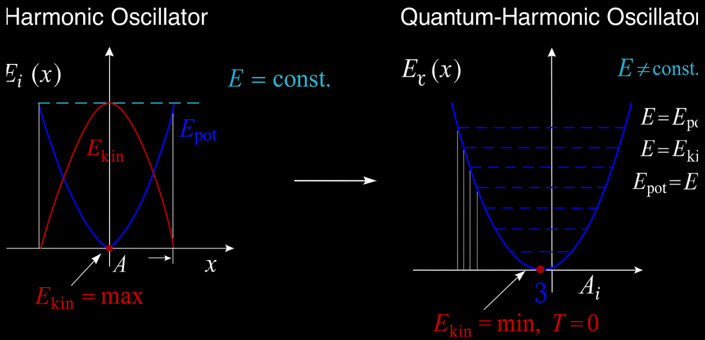
$$x(t) = A\cos(\omega t + \phi)$$

$$E = \frac{1}{2}kA^2 \quad \text{(continuous)}$$



Quantum Harmonic Oscillator: Key Differences

- **Quantized energy**: $E_n = \hbar\omega\left(n + \frac{1}{2}\right)$
- **Probability densities**: Non-uniform even in ground state
- **Zero-point energy**: $E_0 = \frac{1}{2}\hbar\omega \neq 0$
- **Tunneling**: Non-zero probability in classically forbidden regions



Time-Independent Schrödinger Equation

Hamiltonian operator:

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2$$

TISE:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi = E\psi$$

Dimensionless form:

$$\xi = \sqrt{\frac{m\omega}{\hbar}}x, \quad \lambda = \frac{2E}{\hbar\omega}$$

$$\frac{d^2\psi}{d\xi^2} + (\lambda - \xi^2)\psi = 0$$

Asymptotic Behavior and Series Solution

As $\xi \to \pm\infty$:

$$\frac{d^2\psi}{d\xi^2} \approx \xi^2\psi \implies \psi \sim e^{-\xi^2/2}$$

Full solution ansatz:

$$\psi(\xi) = H(\xi)e^{-\xi^2/2}$$

Hermite equation:

$$\frac{d^2H}{d\xi^2} - 2\xi\frac{dH}{d\xi} + (\lambda - 1)H = 0$$

For **normalizable** solutions:

$$\lambda - 1 = 2n \implies \boxed{E_n = \hbar\omega\left(n + \frac{1}{2}\right)}$$

Hermite Polynomials

Solutions to Hermite equation:

$$H_n(\xi) = (-1)^n e^{\xi^2} \frac{d^n}{d\xi^n}(e^{-\xi^2})$$

First few polynomials:

$$\begin{aligned} H_0(\xi) &= 1 \\ H_1(\xi) &= 2\xi \\ H_2(\xi) &= 4\xi^2 - 2 \\ H_3(\xi) &= 8\xi^3 - 12\xi \end{aligned}$$

Orthogonality:

$$\int_{-\infty}^{\infty} H_m(\xi) H_n(\xi) e^{-\xi^2} d\xi = \sqrt{\pi} 2^n n! \delta_{mn}$$

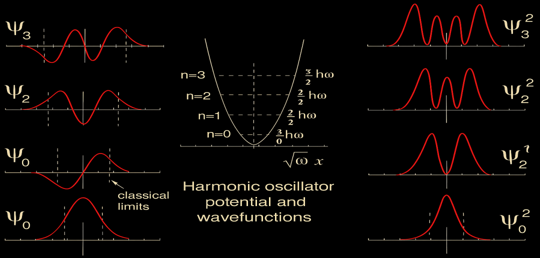
Wavefunctions

Normalized eigenfunctions:

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$$

Properties:

- **Parity**: $\psi_n(-x) = (-1)^n \psi_n(x)$
- **Nodes**: n nodes in ψ_n
- **Orthonormality**: $\int_{-\infty}^{\infty} \psi_m^* \psi_n dx = \delta_{mn}$



Probability Densities

$$P_n(x) = |\psi_n(x)|^2$$

- **Classical limit** ($n \to \infty$): Quantum $\to$ classical distribution
- **Tunneling**: Non-zero probability beyond classical turning points

Zero-Point Energy

$$E_0 = \frac{1}{2}\hbar\omega$$

Consequences:

- Violates classical prediction ($E_{\text{min}} = 0$)
- Heisenberg uncertainty principle: $\Delta x \Delta p \geq \hbar/2$
- Observable effects: Liquid helium doesn't freeze at atmospheric pressure, Casimir effect

Proof via Uncertainty Principle

Minimize $\langle H\rangle = \frac{\langle p^2\rangle}{2m} + \frac{1}{2}m\omega^2\langle x^2\rangle$
Subject to $\Delta x \Delta p = \hbar/2$
Yields $E_{\text{min}} = \frac{1}{2}\hbar\omega$

Molecular Vibrations

- Diatomic molecule potential approximated as parabolic near minimum
- Vibrational energy levels:

$$E_v = \hbar\omega_e \left(v + \frac{1}{2}\right)$$

- Selection rule: $\Delta v = \pm 1$
- IR spectroscopy: Fundamental transition $\omega = \omega_e$

Quantum Optics

- Electromagnetic field as collection of harmonic oscillators
- Quantization: Photon number states $|n\rangle$
- Zero-point field: Responsible for spontaneous emission
- Squeezed states: Below standard quantum limit in one quadrature

Solid-State Physics

- Phonons: Quantized lattice vibrations
- Debye model: Heat capacity $C_V \propto T^3$ at low T
- Superconductivity: Electron-phonon coupling
- Mössbauer effect: Recoilless gamma-ray emission

Problem 1: Ground State Properties

Problem: For ground state ($n = 0$) of QHO:

1. Verify normalization
2. Compute $\langle x \rangle$ and $\langle p \rangle$
3. Compute $\Delta x \Delta p$

Solution: 1. Wavefunction: $\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-m\omega x^2/2\hbar}$
Normalization:

$$\int_{-\infty}^{\infty} |\psi_0|^2 dx = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \int e^{-m\omega x^2/\hbar} dx = 1$$

2. $\langle x \rangle = 0$ (symmetric), $\langle p \rangle = 0$ (real wavefunction)
3. Compute $\langle x^2 \rangle$ and $\langle p^2 \rangle$:

$$\langle x^2 \rangle = \frac{\hbar}{2m\omega}, \quad \langle p^2 \rangle = \frac{m\omega\hbar}{2}$$

$$\Delta x \Delta p = \sqrt{\langle x^2 \rangle \langle p^2 \rangle} = \frac{\hbar}{2}$$

Problem 2: Transition Probability

Problem: A QHO in $n = 1$ state is exposed to electric field $E(t) = E_0 e^{-t^2/\tau^2}$. Find probability to transition to $n = 0$ state.

Solution: Dipole moment operator: $\hat{d} = -e\hat{x}$

Matrix element:

$$\langle 0|\hat{x}|1\rangle = \sqrt{\frac{\hbar}{2m\omega}}\delta_{01}$$

Fermi's Golden Rule:

$$P_{0\leftarrow 1} = \frac{1}{\hbar^2}\left|\int_{-\infty}^{\infty} E(t)e^{i\omega_{01}t}dt\langle 0|\hat{d}|1\rangle\right|^2$$

Gaussian integral:

$$P = \frac{e^2 E_0^2 \pi\tau^2}{2m\omega\hbar}e^{-\omega^2\tau^2/2}$$

Problem 3: Coherent States

Problem: Show that coherent state $|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$ minimizes $\Delta x \Delta p$.

Solution: 1. Define $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$

2. Compute $\langle x \rangle = \sqrt{\frac{2\hbar}{m\omega}} \text{Re}(\alpha)$

$\langle p \rangle = \sqrt{2m\omega\hbar} \text{Im}(\alpha)$

3. Compute variances:

$$(\Delta x)^2 = \frac{\hbar}{2m\omega}, \quad (\Delta p)^2 = \frac{m\omega\hbar}{2}$$

4. Product:

$$\Delta x \Delta p = \frac{\hbar}{2}$$

Exactly the ground state uncertainty!

Summary

- QHO is exactly solvable with quantized energies $E_n = \hbar\omega(n + 1/2)$
- Wavefunctions involve Hermite polynomials with Gaussian decay
- Zero-point energy is fundamental consequence of quantum mechanics
- Applications span molecular physics, quantum optics, and condensed matter
- Mathematical structure extends to quantum field theory

Key Takeaways

- Universality of QHO makes it essential for quantum physics
- Algebraic methods (ladder operators) provide powerful alternative
- Experimental realizations continue to test quantum foundations

References

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